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Pure Spinor Vertex Operators in Siegel Gauge and Loop Amplitude Regularization

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Abstract

Since the b ghost in the pure spinor formalism is a composite operator depending on non-minimal variables, it is not trivial to impose the Siegel gauge condition $b_0 V = 0$ on BRST-invariant vertex operators. Using the antifield vertex operator V^* of ghost-number $+2$, we show that Siegel gauge unintegrated vertex operators can be constructed as $b_0 V^*$ and Siegel gauge integrated vertex operators as $\int dz b_{-1} b_0 V^*$.

These Siegel gauge vertex operators depend on the non-minimal variables, so scattering amplitudes involving these operators need to be regularized using the prescription developed previously with Nekrasov. As an example of this regularization prescription, we compute the four-point one-loop amplitude with four Siegel gauge integrated vertex operators. This is the first one-loop computation in the pure spinor formalism that does not require unintegrated vertex operators.

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1 Introduction

The pure spinor formalism [1] is a manifestly super-Poincaré covariant description of the superstring which has been successfully used to compute multiloop scattering amplitudes and covariantly quantize Ramond-Ramond backgrounds. One of the most surprising features of this formalism is that the b ghost is not a fundamental worldsheet variable but is a composite operator. Nevertheless, after replacing the b ghost with this composite operator, the rules for computing scattering amplitudes are essentially the same as in bosonic string theory.

In bosonic string theory, it is well-known that when a vertex operator V is in Siegel gauge, i.e. when $b_0 V = 0$, the vertex operator is a conformal primary field. Since the scattering amplitude prescription simplifies when the vertex operators are primary, Siegel gauge is a convenient gauge choice. Furthermore, Siegel gauge is a useful gauge choice in bosonic open string field theory since it reduces the kinetic term $\langle \Phi Q \Phi \rangle$ to $\langle \Phi c_0 L_0 \Phi \rangle$ so that the propagator is simply $\frac{b_0}{L_0}$.

Because the composite operator for the b ghost depends on the non-minimal variables in the pure spinor formalism, it is not immediately obvious how to construct BRST-invariant vertex operators V satisfying the Siegel gauge condition $b_0 V = 0$. For example, the massless open string vertex operator in “minimal” gauge is $V = \lambda^\alpha A_\alpha(x, \theta)$, which does not satisfy $b_0 V = 0$ for any choice of A_α . Note that $V = \lambda^\alpha A_\alpha(x, \theta)$ is a conformal primary whenever $\square A_\alpha = 0$ (which implies Lorentz gauge for the gluon). So the condition $b_0 V = 0$ in the pure spinor formalism implies more than just the condition that V is primary.

As will be shown here, a natural way to construct vertex operators in Siegel gauge is to start with the vertex operator V^* for the antifield which has ghost-number two. One then flips the statistics of the antifield by defining V^* to be a bosonic operator. Finally, one constructs the unintegrated ghost-number one vertex operator V_S in Siegel gauge as $V_S = b_0 V^*$. The corresponding integrated ghost-number zero vertex operator in Siegel gauge is $\int dz b_{-1} b_0 V^*$.

This construction in bosonic string theory obviously reproduces the usual Siegel gauge vertex operators where $V^* = c_0 V_S$. But in the pure spinor formalism where there is no c ghost, this construction of Siegel gauge vertex operators is less trivial. For example, since the composite operator for b depends on the non-minimal variables, the resulting Siegel gauge vertex operator V_S will also depend on the non-minimal variables.

The simplest example is the Siegel gauge massless vertex operator which is constructed from the ghost-number two operator $V^* = \lambda^\alpha \lambda^\beta A_{\alpha\beta}(x, \theta)$ where $A_{\alpha\beta}(x, \theta)$ is a bispinor superfield. As shown in [2], $QV^* = 0$ and $\delta V^* = Q(\lambda^\alpha \Omega_\alpha)$ implies that the component fields in $A_{\alpha\beta}(x, \theta)$ describe the antifields to the super-Yang-Mills gluon and gluino. The corresponding Siegel gauge vertex operator for the super-Yang-Mills multiplet is constructed

in unintegrated form as $b_0 V^*$, and in integrated form as $\int dz b_{-1} b_0 V^*$.

Since the composite operator for the b ghost contains poles when the pure spinor variable satisfies $\lambda^\alpha = 0$, the vertex operator in Siegel gauge will also contain these poles. As explained in [3], these poles cause the functional integration over λ^α to diverge if the order of the poles is greater than or equal to 11. However, whenever this divergence occurs, the functional integral over the fermionic non-minimal variables will vanish. The resulting 0/0 ambiguity can be regularized in a BRST-invariant manner using the regularization prescription developed with Nikita Nekrasov in [4].

In this paper, we shall review how this regularization prescription works for general multiloop scattering amplitudes. Furthermore, we will give the first non-trivial application of this regularization procedure by computing a 4-point one-loop amplitude when all four vertex operators are chosen in Siegel gauge in integrated form. Note that, as in bosonic string theory, N -point one-loop amplitudes can be computed using N integrated vertex operators only if all of the vertex operators are in Siegel gauge. So all previous one-loop computations using the pure spinor formalism required at least one unintegrated vertex operator.

It is possible that this new one-loop amplitude prescription will be useful for comparing with the operator approach in the pure spinor formalism, or with other superstring prescriptions such as the Lee-Siegel [5] or RNS prescriptions. Another possible application of our results is for super-Poincaré covariant open superstring field theory. Although a cubic open superstring field theory action has been constructed using the pure spinor formalism [6], the gauge-fixing of this action has not yet been performed. It seems likely that the gauge-fixing techniques developed here will also be useful for gauge-fixing the open superstring field theory action.

This paper is organized as follows. In section 2, the basics of the pure spinor formalism is reviewed and, in section 3, the regularization method proposed in [4] to deal with $(\bar{\lambda}\lambda)$ poles is explained. Section 4 deals with the construction of the vertex operators in the Siegel gauge. Both unintegrated and integrated vertex operators are described, and some of their properties are studied. In section 5, the Siegel gauge vertex operators are used to define a new n -point 1-loop amplitude prescription that uses only integrated vertex operators. In addition, the regularization of [4] is explained using the example of the 4-point 1-loop amplitude. We conclude in section 6 and indicate some directions for future works.

Note: While this paper was being written up, we received a draft of a paper by P.A. Grassi and P. Vanhove which also discusses Siegel gauge vertex operators and regularization in the pure spinor formalism. However, their discussion differs considerably from our paper. At the end of section 3, we have added some comments related to their paper [7] which appeared shortly after the original version of this paper.

2 A review of the pure spinor formalism

We begin by reviewing certain aspects of the pure spinor formalism that are relevant to the present paper.

2.1 World sheet fields

Field contents of the worldsheet theory of the pure spinor formalism can be divided into matter and ghost sectors. The former consists of the Green-Schwarz-Siegel variables [8]

$$(x^m; p_\alpha, \theta^\alpha), \quad m = 0, \dots, 9, \quad \alpha = 1, \dots, 16 \quad (2.1)$$

that describe the embedding of the string in a superspace (x^m, θ^α) .¹ They satisfy free field OPEs

$$x^m(z)x^n(w) = -\eta^{mn} \log(z-w), \quad p_\alpha(z)\theta^\beta(w) = \frac{\delta_\alpha^\beta}{z-w}. \quad (2.2)$$

The ghost sector consists of a conjugate pair of bosonic spinors

$$(\omega_\alpha, \lambda^\alpha), \quad (2.3)$$

but they must be treated with care as they are not genuine free fields; instead, λ^α (weight 0) is subject to the so-called pure spinor constraint

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0, \quad (2.4)$$

where γ_α^m is the symmetric 16×16 gamma matrices in ten dimensions.

To be consistent with this constraint, the conjugate ω_α (weight 1) is defined up to a gauge transformation

$$\delta\omega_\alpha(z) = \Omega_m(z)(\gamma^m\lambda)_\alpha, \quad (2.5)$$

and ω_α can only appear in gauge invariant combinations. Some basic invariants of this gauge transformation are λ -charge current J_λ , Lorentz generator N_{mn} and energy-momentum tensor T_λ defined as

$$J_\lambda = \omega\lambda, \quad N_{mn} = \frac{1}{2}(\omega\gamma_{mn}\lambda), \quad T_\lambda = \omega\partial\lambda. \quad (2.6)$$

¹Throughout, we shall use the notation appropriate for describing a chiral half of the closed string theory, but use a terminology appropriate for the open string.

The OPE algebra formed by those basic gauge invariants can be computed by parameterizing the components of λ^α and ω_α by $U(5)$ covariant genuine free fields. The resulting algebra is

$$\begin{aligned}
N^{mn}(z)\lambda^\alpha(w) &= \frac{\frac{1}{2}(\gamma^{mn}\lambda)^\alpha(w)}{z-w}, \quad J_\lambda(z)\lambda^\alpha(w) = \frac{\lambda^\alpha(w)}{z-w}, \\
N^{mn}(z)N^{pq}(w) &= \frac{-3(\eta^{np}\gamma^{mq} - \eta^{mp}\gamma^{nq})}{(z-w)^2} + \frac{\eta^{np}N^{mq}(w) + (3\text{-terms})}{z-w}, \\
J_\lambda(z)J_\lambda(w) &= \frac{-4}{(z-w)^2}, \quad J_\lambda(z)N^{mn}(w) = \text{regular}, \\
N_{mn}(z)T_\lambda(w) &= \frac{N_{mn}(w)}{(z-w)}, \quad J_\lambda(z)T_\lambda(w) = \frac{-8}{(z-w)^3} + \frac{J_\lambda(w)}{(z-w)^2}, \\
T_\lambda(z)T_\lambda(w) &= \frac{11}{(z-w)^4} + \frac{2T_\lambda(w)}{(z-w)^2} + \frac{\partial T_\lambda(w)}{z-w}.
\end{aligned} \tag{2.7}$$

2.2 Physical states

Physical open string states are defined as the ghost number (λ -charge) 1 cohomology of a nilpotent BRST operator

$$Q = \oint \lambda^\alpha d_\alpha \tag{2.8}$$

where

$$d_\alpha = p_\alpha + \frac{1}{2}(\gamma_m \theta)_\alpha \partial x^m - \frac{1}{8}(\gamma_m \theta)_\alpha (\theta \gamma^m \partial \theta) \tag{2.9}$$

has the form of the phase space constraint of the classical Green-Schwarz action. Using the free field OPE between p_α and θ^α , d_α satisfies

$$d_\alpha(z)d_\beta(w) = \frac{\Pi_m \gamma_{\alpha\beta}^m}{z-w} \tag{2.10}$$

where $\Pi_m = \partial x_m + \frac{1}{2}(\theta \gamma^m \partial \theta)$ is the supersymmetric momentum. d_α and Π_m acts on superfields as supercovariant derivatives:

$$D_\alpha = \partial_\alpha - \frac{1}{2}(\gamma^m \theta)_\alpha \partial_m, \quad P_m = -\partial_m. \tag{2.11}$$

For example, massless states are described by the λ -charge 1 vertex operator

$$V = \lambda^\alpha A_\alpha(x, \theta). \tag{2.12}$$

The cohomology condition $QV = 0$ and $\delta V = Q\Omega$ implies that the superfield A_α satisfies the correct on-shell constraint $\gamma_{m_1 \dots m_5}^{\alpha\beta} D_\alpha A_\beta = 0$ and gauge invariance $\delta A_\alpha = D_\alpha \Omega$.

Although the form of the BRST operator above appears strange at first sight, its ghost number 1 cohomology can be explicitly studied using an $SO(8)$ parameterization of pure spinors and it reproduces the lightcone spectrum of the Green-Schwarz superstring [9]. Moreover, there are arguments that it can in fact be derived from the classical Green-Schwarz action [10, 11, 12, 13, 14], and that it is related to the BRST operators of Ramond-Neveu-Schwarz and Green-Schwarz formalisms by similarity transformations [16].

Finally, Q has cohomologies at other λ -charges as well. They are interpreted as space-time ghosts, antifields and antighosts.

2.3 Pure spinor sector as a curved $\beta\gamma$ system: non-minimal formalism

At first sight, handling a non-linearly constrained system such as the pure spinor system appears difficult. However, it can be treated rigorously using the theory of curved $\beta\gamma$ -systems [17] (if the origin $\lambda \equiv 0$ of the pure spinor space is removed [20]). One way to apply this idea to the pure spinor formalism is to introduce another set of pure spinors and its fermionic partners, $(\bar{\omega}^\alpha, \bar{\lambda}_\alpha; s^\alpha, r_\alpha)$.

$\bar{\lambda}_\alpha$ is an antichiral pure spinor (weight 0)

$$\bar{\lambda}_\alpha \gamma_m^{\alpha\beta} \bar{\lambda}_\beta = 0, \quad (2.13)$$

r_α is a fermionic field (weight 0) that is constrained as

$$r_\alpha \gamma_m^{\alpha\beta} \bar{\lambda}_\beta = 0, \quad (2.14)$$

and $\bar{\omega}^\alpha$ and s^α are the conjugate momenta (weight 1) of $\bar{\lambda}_\alpha$ and r_α , respectively. In ten dimensional Euclidean space, $\bar{\lambda}_\alpha$ can be regarded as the complex conjugate of λ^α , and r_α is an extension of the target space differential of $\bar{\lambda}_\alpha$:

$$r_\alpha \sim d\bar{\lambda}_\alpha. \quad (2.15)$$

Just as ω_α must appear in invariant combinations under the gauge transformation $\delta\omega_\alpha = \Omega_m(\gamma^m \lambda)_\alpha$, the conjugates $\bar{\omega}^\alpha$ and s^α must appear in invariant combinations under

$$\delta\bar{\omega}^\alpha = \bar{\Omega}_m(\gamma^m \bar{\lambda})^\alpha - \phi_m(\gamma^m r)^\alpha, \quad \delta s^\alpha = \phi_m(\gamma^m \bar{\lambda})^\alpha, \quad (2.16)$$

for arbitrary $\bar{\Omega}_m$ and ϕ_m . Some basic invariants are

$$\begin{aligned} \bar{N}_{mn} &= \frac{1}{2}(\bar{\omega}\gamma_{mn}\lambda - s\gamma_{mn}r), \quad \bar{J}_{\bar{\lambda}} = \bar{\omega}\bar{\lambda} - sr, \quad T_{\bar{\lambda}} = \bar{\omega}\partial\bar{\lambda} - s\partial r, \\ J_r &= -sr, \quad S_{mn} = \frac{1}{2}(s\gamma_{mn}\bar{\lambda}), \quad S = s\bar{\lambda}. \end{aligned} \quad (2.17)$$

By parameterizing non-minimal variables by $U(5)$ covariant free fields (antiholomorphic local coordinates on the pure spinor space and their conjugates), OPE's among the basic invariants can be computed. In particular, they satisfy

$$\begin{aligned}
J_r(z)J_r(w) &= \frac{11}{(z-w)^2}, \quad \bar{J}_{\bar{\lambda}}(z)J_r(w) = \frac{8}{(z-w)^2}, \quad \bar{J}_{\bar{\lambda}}(z)\bar{J}_{\bar{\lambda}}(w) = \text{regular}, \\
J_r(z)T_{\bar{\lambda}}(w) &= \frac{11}{(z-w)^3} + \frac{J_r(w)}{(z-w)^2}, \quad \bar{J}_{\bar{\lambda}}(z)T_{\bar{\lambda}}(w) = \frac{\bar{J}_{\bar{\lambda}}}{(z-w)^2}, \\
T_{\bar{\lambda}}(z)T_{\bar{\lambda}}(w) &= \frac{2T_{\bar{\lambda}}(w)}{(z-w)^2} + \frac{\partial T_{\bar{\lambda}}(w)}{z-w}.
\end{aligned} \tag{2.18}$$

Therefore, the addition of the non-minimal sector does not affect the total central charge, but the total ghost number anomaly is shifted to $3 = 11 - 8$, if one defines the ghost number by

$$J_g = J_{\lambda} - J_r - \bar{J}_{\bar{\lambda}} = \omega\lambda - \bar{\omega}\bar{\lambda}. \tag{2.19}$$

Physical states are then redefined as the ghost number 1 cohomology of a nilpotent BRST operator

$$\begin{aligned}
Q &= Q_0 + Q_1 \\
Q_0 &= \oint \lambda^{\alpha} d_{\alpha}, \quad Q_1 = \oint r_{\alpha} \bar{\omega}^{\alpha}.
\end{aligned} \tag{2.20}$$

Subscripts denote the r -charge, but note that both Q_0 and Q_1 carry +1 charge under the ghost number current $J_g = \omega\lambda - \bar{\omega}\bar{\lambda}$.

The additional piece Q_1 of the BRST operator deals with the constrained nature of the pure spinor, and is essential for having a composite b -ghost operator that satisfies

$$T = \{Q, b\}, \tag{2.21}$$

for the total energy-momentum tensor T . For future reference, we here record the explicit form of the b -ghost:

$$\begin{aligned}
b &= b_{-1} + b_0 + b_1 + b_2 + b_3 \\
b_{-1} &= s^{\alpha} \partial \bar{\lambda}_{\alpha} \\
b_0 &= \frac{\bar{\lambda}_{\alpha} [2\Pi^m (\gamma_m d)^{\alpha} - N_{mn} (\gamma^{mn} \partial \theta)^{\alpha} - J_{\lambda} \partial \theta^{\alpha} - \frac{1}{4} \partial^2 \theta^{\alpha}]}{4(\bar{\lambda}\lambda)} \\
b_1 &= \frac{(\bar{\lambda} \gamma^{mnp} r)(d \gamma_{mnp} d + 24 N_{mn} \Pi_p)}{192(\bar{\lambda}\lambda)^2} \\
b_2 &= \frac{(r \gamma_{mnp} r)(\bar{\lambda} \gamma^m d) N^{np}}{16(\bar{\lambda}\lambda)^3} \\
b_3 &= \frac{(\bar{\lambda} \gamma^{mnp} r)(r \gamma_p^{qr} r) N_{mn} N_{qr}}{128(\bar{\lambda}\lambda)^4}.
\end{aligned} \tag{2.22}$$

Now, Q_1 itself is nilpotent and its cohomology can be regarded as the operator space of the pure spinor sector. In [23, 24, 25], the structure of this operator space was studied by computing its partition function. One outcome of the investigation was that Q_1 -cohomology consists of two sectors $H^0(Q_1)$ and $H^3(Q_1)$, and there is a one-to-one mapping between the two. (Here, the degree of the cohomology is the differential form degree carried by $r_\alpha \sim d\bar{\lambda}_\alpha$.) An important element of $H^3(Q_1)$ is the tail term b_3 of the composite b -ghost. It was found that $H^3(Q_1)$ is essential for having the operator doubling between spacetime fields and antifields, and found that the total cohomology $H^*(Q_1)$ has precisely the right structure to kill the unphysical degrees of freedom contained in the covariant oscillators $(x^m, p_\alpha, \theta^\alpha)$, up to the fifth mass level.

Having introduced the basic ingredients of the pure spinor formalism, we now turn to the description of scattering amplitudes.

2.4 Tree amplitude

To compute n -point tree amplitudes, one uses 3 unintegrated vertex operators V and $n-3$ integrated vertex operators $\int dz U(z)$ as in the bosonic string, where U carries weight 1, ghost number 0, and is related to V as $QU = \partial V$:

$$\mathcal{A}_n = \int \prod_{i=4}^n d^2 w_i |\langle V_1(z_1) V_2(z_2) V_3(z_3) \prod_{i=4}^n U(w_i) \rangle|^2 \quad (2.23)$$

Here, $\langle \dots \rangle$ denotes functional integrations over $(\theta^\alpha, \lambda^\alpha, \bar{\lambda}_\alpha, r_\alpha)$. (We ignore the functional integration over x^m for simplicity because there is nothing special about it in the pure spinor formalism.)

After integrating out the non-zero modes using OPE's, one is left with the zero-mode integration of the form

$$\mathcal{A} = \int [d\lambda][d\bar{\lambda}][dr] d^{16}\theta \mathcal{N} f(\lambda, \bar{\lambda}, r, \theta), \quad (2.24)$$

where $f(\lambda, \bar{\lambda}, r, \theta)$ is a function of the zero modes, and the zero mode measures behave as

$$[D\lambda] = \lambda^{-3} d^{11}\lambda, \quad [D\bar{\lambda}] = \bar{\lambda}^{-3} d^{11}\bar{\lambda}, \quad [Dr] = \bar{\lambda}^3 d^{11}r. \quad (2.25)$$

Now, this zero mode integration is ambiguous due to an indefinite factor $\infty \cdot 0$ that comes from non-compact bosonic integration over λ^α and $\bar{\lambda}_\alpha$, and unsaturated fermionic integration over θ^α and r_α . However, this difficulty can be easily overcome by inserting a regularization factor of the form

$$\mathcal{N}_0 = \exp\{Q, \chi\}. \quad (2.26)$$

Since $\mathcal{N}_0 = 1 + Q\Omega$ for some Ω , \mathcal{N}_0 respects BRST symmetry, and one is free to choose whatever χ that is convenient for computing amplitudes. A simple and convenient choice is

$$\chi = -\bar{\lambda}\theta, \quad (2.27)$$

which leads to

$$\mathcal{N}_0 = \exp(-\bar{\lambda}\lambda - r\theta). \quad (2.28)$$

\mathcal{N}_0 puts an exponential cut-off for bosonic zero-modes, and provides extra fermionic zero-modes via an expansion of the exponential. By now, it is well tested that this prescription lead to correct tree amplitudes (see for example [26] for a review).

2.5 Loop amplitudes

We now turn to the discussion of loop amplitudes. A natural prescription to use for the n -point g -loop amplitude is

$$\mathcal{A}_g = \int_{\mathcal{M}_{g,n}} d^{3g-3} \tau_k \prod_{i=1}^n \int d^2 w_i \prod_{k=1}^{3g-3} \int d^2 z_k |\langle \prod_{k=1}^{3g-3} (\mu_k \cdot b)(z_k) \prod_{i=1}^n U(w_i) \rangle|^2, \quad (2.29)$$

where τ_k and μ_k are the Teichmüller parameters and associated Beltrami operator, and the bracket $\langle \dots \rangle$ denotes functional integrations over the worldsheet fields. However, this prescription is incomplete as there are two subtleties with the functional integration over zero modes.

The first subtlety comes from the proper definition of an $\infty \cdot 0$ factor associated with the integration over non-compact bosonic zero modes, and over unsaturated fermionic zero modes. This indefinite factor can be defined as before by inserting an operator of the form $\mathcal{N}_0 = \exp\{Q, \chi_0\}$ to the zero mode integral. The only modification needed is that now there are zero modes for weight 1 operators $(N_{mn}, \bar{N}_{mn}, J_\lambda, \bar{J}_\lambda, S_{mn}, S)$ as well.

A convenient choice for χ_0 is

$$\chi_0 = -\bar{\lambda}\theta - \frac{1}{2} \sum_{I=1}^g (N_{I,mn} S_I^{mn} + J_I S_I) \quad (2.30)$$

where $(N_{I,mn}, J_I; S_{I,mn}, S_I)_{I=1 \sim g}$ are the g zero modes of the corresponding operators defined in (2.17). For this choice, the zero mode integration comes with an insertion of

$$\begin{aligned} \mathcal{N}_0 &= \exp[-(\bar{\lambda}\lambda + r\theta)] \\ &\times \exp \sum_{I=1}^g \left[-\frac{1}{2} N_{I,mn} \bar{N}_I^{mn} - J_I \bar{J}_I - \frac{1}{2} S_{I,mn} (d_I \gamma^{mn} \lambda) - S_I (\lambda d_I) \right]. \end{aligned} \quad (2.31)$$

\mathcal{N}_0 puts an exponential cut-off for bosonic zero-modes, and provides extra fermionic zero-modes via an expansion of the exponential.

There is another (possible) source of an indefinite factor $\infty \cdot 0$ coming from the integration around $(\bar{\lambda}\lambda) \sim 0$. This second subtlety is due to the $(\bar{\lambda}\lambda)$ poles in the integrand that could come from the measure, from the insertion of the composite b -ghosts, and from the vertex operators. When the order of the $(\bar{\lambda}\lambda)$ pole is too high, one gets a divergence upon integrating near $(\bar{\lambda}\lambda) \sim 0$. However, this type of divergence always comes with a zero coming from an over saturation of r zero modes, and it was shown in [4] how to regularize and define this indefinite factor.

Since this regularization for $(\bar{\lambda}\lambda) \sim 0$ is slightly involved, we shall explain it in a separate (next) section. In the absence of the second subtlety at $(\bar{\lambda}\lambda) \rightarrow 0$, the prescription above is well tested and known to lead to correct loop amplitudes [27]. (See [26] for a review on the subject, and [28] for an extension of loop computations to eleven dimensions.)

3 Regularization of $(\bar{\lambda}\lambda) \sim 0$

In this section, we explain the regularization prescription of [4] for the functional integral region $(\bar{\lambda}\lambda) \sim 0$. Although the basic idea of [4] is simple, the formulas there ended up complicated because one had to make the prescription consistent with the pure spinor constraint (or more specifically, with the gauge invariance under $\delta\omega_\alpha = (\gamma_m\lambda)_\alpha\Omega^m$). In order to demonstrate how the prescription regularizes the region $(\bar{\lambda}\lambda) \sim 0$, we here ignore the subtleties coming from the pure spinor constraint.

3.1 Terms requiring regularization of $(\bar{\lambda}\lambda) \sim 0$

To estimate the order of divergence as $(\bar{\lambda}\lambda)$ approaches 0, it is more convenient to use ω_α and $\bar{\omega}^\alpha$ instead of their gauge invariant counterparts, (N_{mn}, J_λ) and $(\bar{N}_{mn}, \bar{J}_\lambda)$. On a genus g surface, the zero mode integration measures for the pure spinor variables behave as [6]

$$[D\lambda] = \lambda^{-3} d^{11}\lambda, \quad [DN] = \lambda^{-8g} d^g J d^{10g} N \rightarrow [D\omega] = \lambda^{3g} d^{11g}\omega, \quad (3.1)$$

$$[D\bar{\lambda}] = \bar{\lambda}^{-3} d^{11}\bar{\lambda}, \quad [D\bar{\omega}] = \bar{\lambda}^{3g} d^{11g}\bar{\omega}, \quad [Dr] = \bar{\lambda}^3 d^{11}r, \quad [Ds] = \bar{\lambda}^{-3g} d^{11g}s. \quad (3.2)$$

The total measure thus goes as

$$[D\lambda D\bar{\lambda} D\omega D\bar{\omega} Dr Ds] = \lambda^{3g-3} d^{22}\lambda d^{22g}\omega d^{11}r d^{11g}s \quad (3.3)$$

where we denoted $d^{22}\lambda = d^{11}\lambda d^{11}\bar{\lambda}$ etc.

So when the poles of the various operators in the correlator add up to $\bar{\lambda}^{-11}\lambda^{-3g-8}$ or higher, the zero-mode path integrals over λ^α and $\bar{\lambda}_\alpha$ become ill-defined because

$$\int d^{22}\lambda \frac{1}{(\bar{\lambda}\lambda)^L} \quad (3.4)$$

diverges at $(\bar{\lambda}\lambda) \rightarrow 0$ for $L \geq 11$. Fortunately, it turns out that each factor of $(\bar{\lambda}\lambda)^{-1}$ comes with a factor of r_α , so for $L > 11$ one always gets a zero from over-saturated r_α zero modes as well. (The case of $L = 11$ will be discussed at the end of section 3.2.) To see this, note that on a genus g surface, one needs $3g - 3$ b ghost insertion and each term in the b ghost goes as $\bar{\lambda}r^{k-1}/(\bar{\lambda}\lambda)^k$. Hence, after combining with the λ^{3g-3} pole from the measure, one indeed gets integration of the form

$$\int d^{22}\lambda d^{11}r \sum_k \frac{r^k}{(\bar{\lambda}\lambda)^k}. \quad (3.5)$$

Therefore, the problem is again to define the integral of

$$\int d^{22}\lambda d^{11}r \frac{r^L}{(\bar{\lambda}\lambda)^L} \sim 0 \cdot \infty, \quad (L \geq 11). \quad (3.6)$$

Using the idea described in [4], we now explain that an appropriate regularization can be done by an insertion of an operator of the form $\mathcal{N}'(y) = \exp\{Q, \chi(y)\} = 1 + Q\Omega(y)$.

3.2 Regularization of $(\bar{\lambda}\lambda) \sim 0$

In order to concentrate on the main idea, we first explain how to regularize the divergent integral (3.4) over λ^α and $\bar{\lambda}_\alpha$, ignoring the zero that comes from r_α integration. The method can be extended to respect the BRST invariance, and then it naturally defines the $0 \cdot \infty$ of (3.6).

The basic idea of [4] is to prevent the $(\bar{\lambda}\lambda)$ poles in operators at different worldsheet positions from diverging simultaneously. This can be achieved by shifting the target-space location of each pole by different constants f_i 's:

$$\prod_{i=1}^n \frac{1}{|\lambda(w_i)|^{2l_i}} \rightarrow \prod_{i=1}^n \frac{1}{|\lambda(w_i) + f_i|^{2l_i}}. \quad (3.7)$$

Then, after integrating out the non-zero modes, one is left with the integrand of the form

$$\prod_{i=1}^n \frac{1}{|\lambda + f_i|^{2l_i}}, \quad \sum l_i = L \quad (3.8)$$

instead of just $(\bar{\lambda}\lambda)^{-L}$. The integration over $(\lambda, \bar{\lambda})$ is now well defined as long as $l_i < 11$ in each factor $|\lambda + f_i|^{-2l_i}$. Eventually, one can average over the constants f_i 's to get a finite result.

To achieve the shifting in the pure spinor formalism in a BRST invariant manner, we introduce a constant pure spinor f^α and its fermionic partner g^α (targetspace differential $\sim df^\alpha$), as well as their complex conjugates \bar{f}_α and \bar{g}_α . They are constrained as

$$f^\alpha \gamma_{\alpha\beta}^m f^\beta = f^\alpha \gamma_{\alpha\beta}^m g^\beta = \bar{f}_\alpha \gamma_m^{\alpha\beta} \bar{f}_\beta = \bar{f}_\alpha \gamma_m^{\alpha\beta} \bar{g}_\beta = 0. \quad (3.9)$$

Then, we extend the BRST operator to

$$Q \rightarrow Q' = Q + f^\alpha \frac{\partial}{\partial g^\alpha} + \bar{g}_\alpha \frac{\partial}{\partial \bar{f}_\alpha} \quad (3.10)$$

and introduce an additional regularization factor of the form

$$\mathcal{N}'(y) = \exp\{Q, \chi(y)\} \quad (3.11)$$

at an arbitrary point y on the worldsheet. To achieve the shift, we include $g^\alpha \omega_\alpha(y) + \bar{f}_\alpha s^\alpha(y)$ in $\chi(y)$, and also put zero modes $\sum_I \omega_{\alpha,I} s_I^\alpha$ to impose an exponential cut-off for the integration over ω_α and $\bar{\omega}^\alpha$.² So we have

$$\begin{aligned} \mathcal{N}'(y) = & \exp\left[-\sum_{I=1}^g (\bar{\omega}_I^\alpha \omega_{\alpha,I} + s_I^\alpha d_{\alpha,I})\right] \\ & \times \exp(f^\alpha \omega_\alpha(y) + g^\alpha d_\alpha(y) + \bar{f}_\alpha \bar{\omega}^\alpha(y) + \bar{g}_\alpha s^\alpha(y)). \end{aligned} \quad (3.12)$$

Unlike the zero mode regulator \mathcal{N}_0 for $(\bar{\lambda}\lambda) \rightarrow \infty$, the regulator $\mathcal{N}'(y)$ includes non-zero modes in an essential way. In particular, non-zero modes of s^α are important for removing extra r_α 's in the integrand.

The easiest way to understand that $\mathcal{N}'(y)$ indeed brings about the desired shift of poles is to use the path integral formalism. On a genus g surface, each component of ω_α and λ^α can be expanded by a complete set of eigenfunctions of the worldsheet Laplacian as:

$$\omega_\alpha(z) = \sum_{I=1}^g \omega_{\alpha,I} \Omega_I(z) + \sum_{I'} \omega_{\alpha,I'} \Omega_{I'}(z, \bar{z}) \quad (3.13)$$

$$\lambda^\alpha(z) = \lambda_0^\alpha \Lambda_0 + \sum_{I'} \lambda_{I'}^\alpha \Lambda_{I'}(z, \bar{z}). \quad (3.14)$$

²In [4], the zero mode cut-off $\exp\{Q, -\bar{f}g\} = \exp(-\bar{f}f - \bar{g}g)$ was included in \mathcal{N}' instead of the cut-off for ω_α and $\bar{\omega}^\alpha$, $\exp\sum(-\bar{\omega}\omega - sd)$. Since we have a factor of $\exp(f\omega + \bar{f}\bar{\omega})$ in \mathcal{N}' as well, both should have the same effect upon integration, but we found it simpler in practice to use our choice.

Here, $\Omega_{\alpha,I}(z)$ ($I = 1 \sim g$) are g zero modes of ω_α , Λ_0 is the zero mode of λ^α , $\Omega_{I'}(z)$ and $\Lambda_{I'}(z)$ are the non-zero modes of ω_α and λ^α . Then, the Green function

$$G(y, z) = \sum_{I'} \Omega_{I'}(y, \bar{y}) \Lambda_{I'}(z, \bar{z}) \quad (3.15)$$

satisfies

$$\partial_{\bar{y}} G(y, z) = \sum_{I'} \Lambda_{I'}^*(y, \bar{y}) \Lambda_{I'}(z, \bar{z}) = \delta^2(z - y) - |\Lambda_0|^2 \quad (3.16)$$

$$\partial_{\bar{z}} G(y, z) = \sum_{I'} \Omega_{I'}(y, \bar{y}) \Omega_{I'}^*(z, \bar{z}) = \delta^2(z - y) - \sum_I \Omega_I(y) \Omega_I^*(z). \quad (3.17)$$

Now, using the Green function $G(y, z)$, $\mathcal{N}'(y)$ can be rewritten as

$$\begin{aligned} \mathcal{N}'(y) &= \exp\left[-\sum_{I=1}^g (\bar{\omega}_I^\alpha \omega_{\alpha,I} + s_I^\alpha d_{\alpha,I})\right] \\ &\quad \times \exp \int d^2 z \delta^2(y - z) (f^\alpha \omega_\alpha + g^\alpha d_\alpha + \bar{f}_\alpha \bar{\omega}^\alpha + \bar{g}_\alpha s^\alpha)(z) \end{aligned} \quad (3.18)$$

$$= \mathcal{N}'_0 \times \exp \int d^2 z \partial_{\bar{z}} G(y, z) (f^\alpha \omega_\alpha + g^\alpha d_\alpha + \bar{f}_\alpha \bar{\omega}^\alpha + \bar{g}_\alpha s^\alpha)(z), \quad (3.19)$$

where

$$\mathcal{N}'_0 = \exp \sum_{I=1}^g [-(\bar{\omega}_{\alpha,I} \omega_I^\alpha + s_I^\alpha d_{\alpha,I}) + (f^\alpha \omega_{\alpha,I} + g^\alpha d_{\alpha,I} + \bar{f}_\alpha \bar{\omega}_I^\alpha + \bar{g}_\alpha s_I^\alpha)], \quad (3.20)$$

and $(\omega_{\alpha,I}, \bar{\omega}_I^\alpha, d_{\alpha,I}, s_I^\alpha)$ are the g zero modes of $(\omega_\alpha, \bar{\omega}^\alpha, d_\alpha, s^\alpha)$. Therefore, except for the zero mode factor \mathcal{N}'_0 , insertion of $\mathcal{N}'(y)$ in the path integral can be absorbed into the change of variables³

$$\begin{aligned} \lambda'^\alpha(z) &= \lambda^\alpha(z) + f^\alpha G(y, z), \quad \bar{\lambda}'_\alpha(z) = \bar{\lambda}_\alpha(z) + \bar{f}_\alpha G(y, z), \\ \theta'^\alpha(z) &= \theta^\alpha(z) + g^\alpha G(y, z), \quad r'_\alpha(z) = r_\alpha(z) + \bar{g}_\alpha G(y, z), \\ x'^m(z) &= x^m(z) - \frac{1}{2} g^\alpha (\gamma^m \theta(y))_\alpha G(y, z), \quad d'_\alpha(z) = d_\alpha(z) + (g \gamma^m)_\alpha \Pi_m(y) G(y, z), \end{aligned} \quad (3.21)$$

as in

$$\int D\phi \mathcal{N}'(y) \exp(-S) = \int D\phi \mathcal{N}'_0 \exp(-S') \quad (3.22)$$

$$S = \int d^2 z \left(\frac{1}{2} \partial_z x^m \partial_{\bar{z}} x_m + p \partial_{\bar{z}} \theta - \omega \partial_{\bar{z}} \lambda - \bar{\omega} \partial_{\bar{z}} \bar{\lambda} + s \partial_{\bar{z}} r \right). \quad (3.23)$$

³ We thank Joost Hoogeveen for pointing out that x^m must also be included in the change of variables.

In other words, the path integral definition of the correlation functions

$$\langle O_1(w_1)O_2(w_2)\cdots O_n(w_n)\rangle = \int D\phi \exp(-S)\mathcal{N}'(y)O_1(w_1)O_2(w_2)\cdots O_n(w_n) \quad (3.24)$$

is equivalent to

$$\langle O_1(w_1)O_2(w_2)\cdots O_n(w_n)\rangle' = \int D\phi' \exp(-S')\mathcal{N}'_0(y)O'_1(w_1)O'_2(w_2)\cdots O'_n(w_n), \quad (3.25)$$

where in $O'_i(w_i)$, variables $(\lambda^\alpha, \bar{\lambda}_\alpha, \theta^\alpha, r_\alpha)$ are shifted as in (3.21). For example, an operator of the form

$$\frac{r^3 F(x, \theta)}{(\bar{\lambda}\lambda)^3}(w_i), \quad (F: \text{some superfield}) \quad (3.26)$$

gets modified to

$$\frac{(r' + \bar{g}_i)^3 F(x' - \frac{1}{2}g_i\gamma\theta, \theta' + g_i)}{|\lambda' + f_i|^6}(w_i) \quad (3.27)$$

where we abbreviated as

$$f_i^\alpha = f^\alpha G(y, w_i). \quad (3.28)$$

Below, we omit primes from $(\lambda^\alpha, \bar{\lambda}, \theta^\alpha, r_\alpha)$ for simplicity.

So, the computation of a g -loop amplitude typically reduces to a sum of zero mode integrations of the form

$$\int d^{16g}d d^{16}\theta d^{22g}\omega \int d^{22}f d^{22}g \int d^{22}\lambda d^{11}r \mathcal{N}_0 \mathcal{N}'_0 \prod_i \frac{(r + \bar{g}_i)^{l_i} F_i(x - \frac{1}{2}g_i\gamma\theta, \theta + g_i)}{|\lambda + f_i|^{2l_i}}. \quad (3.29)$$

When the total order of divergence $\sum l_i$ is smaller than 11, \mathcal{N}'_0 regularization does nothing and the integral goes back to that of section 2. The case when $\sum l_i = 11$ is special and will be discussed at the end of this subsection. When $11 < \sum l_i < 22$, only the \mathcal{N}'_0 regularization is necessary, and we can set $\mathcal{N}_0 = 1$ using BRST invariance. Moreover, provided each l_i is smaller than 11, one can show that the integral (3.29) is finite (possibly zero due to a lack of some fermionic zero modes).

To prove this, first introduce parameters $(c, \varepsilon, \bar{\varepsilon})$ in \mathcal{N}' as

$$\begin{aligned} \mathcal{N}'(y) = & \exp[-c \sum_I (\bar{\omega}_I^\alpha \omega_I^\alpha + s_I^\alpha d_{\alpha, I})] \\ & \times \exp[\varepsilon(f^\alpha \omega_\alpha(y) + g^\alpha d_\alpha(y) + \bar{\varepsilon}(\bar{f}_\alpha \bar{\omega}^\alpha(y) + \bar{g}_\alpha s^\alpha(y))]. \end{aligned} \quad (3.30)$$

BRST invariance guarantees that the scattering amplitude will be independent of $(c, \varepsilon, \bar{\varepsilon})$. Performing the integration over the zero modes $\omega_{\alpha, I}$ and $\bar{\omega}_I^\alpha$ ($I = 1 \sim g$), this yields an exponential cut-off

$$\exp\left[-\frac{\varepsilon\bar{\varepsilon}}{c}(\bar{f}_\alpha f^\alpha + \bar{g}_\alpha g^\alpha)\right] \quad (3.31)$$

for f^α and \bar{f}_α . Then, it can be shown that the integration over $(\lambda^\alpha, \bar{\lambda}_\alpha, f^\alpha, \bar{f}_\alpha)$ is finite. When $\sum l_i > 11$, it is clear that the only region of integration that can cause divergence is where all $(\lambda^\alpha, \bar{\lambda}_\alpha, f^\alpha, \bar{f}_\alpha)$ become small of order ϵ . But thanks to the smearing by f 's, the integral is now finite in this region for $11 < L = \sum l_i < 22$:

$$\int d^{22}\lambda \frac{1}{|\lambda|^{2L}} \sim \epsilon^{22-2L} \quad \rightarrow \quad \int d^{22}f d^{22}\lambda \frac{1}{|\lambda + f|^{2L}} \sim \epsilon^{44-2L}. \quad (3.32)$$

Although the \mathcal{N}' regularization described so far only works when the order of $(\bar{\lambda}\lambda)$ is below 22, the estimate of the integral above also shows what to do if the order of $(\bar{\lambda}\lambda)$ poles sum up beyond 22. For example, when the total order of divergence L satisfies $22 < L < 33$, one only has to introduce another copy of smearing variables $(f'^\alpha, g'^\alpha, \bar{f}'_\alpha, \bar{g}'_\alpha)$, and extend the regulator $\mathcal{N}'(y)$ so that λ^α gets shifted by both f^α and f'^α . Then, one eventually gets the integral of the form

$$\int d^{22}f d^{22}f' d^{22}\lambda \frac{1}{|\lambda + f + f'|^{2L}} \sim \epsilon^{66-2L}, \quad (3.33)$$

which is finite when $L < 33$.

Finally, we shall discuss the case when $\sum l_i = 11n$ for any positive integer n and will argue that these terms do not contribute to BRST-invariant amplitudes. In this case, an analysis similar to (3.32) and (3.33) implies that integration over the bosonic variables $(\lambda^\alpha, f^\alpha)$ gives a logarithmic dependence on ε . However, integration over the fermionic variables $(\theta^\alpha, g^\alpha)$ can only give polynomial dependence on ε . Since BRST-invariant amplitudes must be independent of ε , this suggests that terms with $\sum l_i = 11n$ cannot contribute to BRST-invariant scattering amplitudes.

To see more explicitly why this happens, consider the term with $\sum l_i = 11$

$$V_{11} \equiv \frac{(\lambda^3 r^{11})}{(\bar{\lambda}\lambda)^{11}} (\theta)^{16} \quad (3.34)$$

where $(\lambda^3 r^{11})$ denotes the unique Lorentz-invariant contraction of three pure spinor λ 's and 11 fermionic r 's. Since the functional integral

$$\int d^{22}\lambda d^{11}r d^{16}\theta \mathcal{N} V_{11} \quad (3.35)$$

is nonzero, such a term would contribute to the scattering amplitude if V_{11} were present in the BRST invariant integrand $f(\lambda, \bar{\lambda}, r, \theta)$ of (2.24).

However, V_{11} cannot appear as part of a BRST-invariant expression for the following reason. First, note that V_{11} is in the cohomology of $Q_1 = \oint r_\alpha \bar{\omega}^\alpha$ and that the only non-trivial cohomology of $Q = Q_0 + Q_1$ at weight 0 and ghost number 3 is $(\lambda^3 \theta^5) \equiv (\lambda \gamma^m \theta)(\lambda \gamma^n \theta)(\lambda \gamma^p \theta)(\theta \gamma_{mnp} \theta)$. So the only possible way for V_{11} to be a part of a BRST invariant operator $\sum_{k=0}^{11} V_k$ is if

$$\sum_{k=0}^{11} V_k = a(\lambda^3 \theta^5) + Q(\sum_{k=0}^{11} \Lambda_k) \quad (3.36)$$

for some set of Λ_k 's, where a is a constant and the subscript k denotes the term proportional to r^k . Since $Q = Q_0 + Q_1$ where $Q_0 = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha}$ and $Q_1 = r_\alpha \frac{\partial}{\partial \lambda_\alpha}$, this would imply that $V_{11} = Q_0 \Lambda_{11} + Q_1 \Lambda_{10}$. Since V_{11} is proportional to $(\theta)^{16}$ and Q_0 lowers the number of θ 's, Λ_{11} must vanish which implies that $V_{11} = Q_1 \Lambda_{10}$. But this cannot happen because V_{11} is in the cohomology of Q_1 . Therefore, the eleventh pole simply does not contribute to the BRST invariant amplitude, and the \mathcal{N}' regularization is unnecessary.

Similarly, the only dangerous term for $\sum_i l_i = 22$ with a single set of smearing variables is

$$V_{22} \equiv \frac{(\lambda^3 r^{11})(\bar{g}g)^{11}(\theta)^{16}}{\prod_i |\lambda + f_i|^{2l_i}} \in \frac{(\lambda)^3(r + \bar{g})^{22}(g)^{11}(\theta)^{16}}{\prod_i |\lambda + f_i|^{2l_i}}. \quad (3.37)$$

Again, one can argue that this is in the cohomology of $Q_1 = r_\alpha \frac{\partial}{\partial \lambda_\alpha} + \bar{g}_\alpha \frac{\partial}{\partial f_\alpha}$ but cannot be part of a BRST invariant operator where now the BRST operator is extended to $Q = Q_0 + Q_1$ with $Q_0 = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + f^\alpha \frac{\partial}{\partial g^\alpha}$. So the twenty-second pole cannot contribute to the BRST invariant amplitude as well.

It is important to stress that the $\mathcal{N}'(y)$ regulator is necessary for resolving 0/0 ambiguities and does not affect the computation of terms with $\sum l_i < 11$ where such ambiguities are not present. Similarly, if one has two copies of $\mathcal{N}'(y_1)\mathcal{N}'(y_2)$ inserted at different points of the worldsheet, the extra regulator does not affect the computation of terms with $\sum l_i < 22$. So for an amplitude computation whose maximum order of divergence is $L = \sum l_i$, it is sufficient to insert $L/11$ regulators at different points on the worldsheet. Of course, one can always insert more than $L/11$ regulators, but the additional regulators will not affect the computation.

It is also important to stress that if one does not include a sufficient number of regulators, the amplitude is ambiguous since one has a divergence (coming from the bosonic functional integral) multiplied by a zero (coming from the fermionic functional integral). In a recent paper [7], Grassi and Vanhove claimed that the amplitude is vanishing if one does not include a sufficient number of regulators, which would lead to a violation of

unitarity. Their claim was based on doing the fermionic functional integral *before* doing the bosonic functional integral, in which case they obtained the zero but ignored the divergence. However, any proper regularization procedure should be independent of the order of integration. So it is necessary to first regularize the divergence in the bosonic functional integral before attempting to do the fermionic functional integral.

Grassi and Vanhove also proposed a new regulator in [7] of the form

$$\hat{\mathcal{N}} = \exp\left[-\frac{1}{\lambda\bar{\lambda}} - r_\alpha\left(\frac{\delta_\beta^\alpha}{(\lambda\bar{\lambda})^2} - 2\frac{\lambda^\alpha\bar{\lambda}_\beta}{(\lambda\bar{\lambda})^3}\right)\theta^\beta\right] \quad (3.38)$$

which removes the divergence when $\lambda\bar{\lambda} \rightarrow 0$. However, their regulator does not remove the divergence when $\lambda\bar{\lambda} \rightarrow \infty$. Although they claim that the divergence when $\lambda\bar{\lambda} \rightarrow \infty$ can be ignored because of the $(\lambda\bar{\lambda})^{-1}$ factors coming from the integration over the r zero modes, this claim is based on the assumption that one can first do the fermionic functional integral before doing the bosonic functional integral. However, as stressed above, any proper regularization procedure must be independent of the order of integration. Since performing the bosonic functional integral *before* performing the integration over the r zero modes leads to an ambiguous answer using the regulator of [7], their regularization procedure is incomplete. Regularization of the bosonic functional integral when $\lambda\bar{\lambda} \rightarrow \infty$ is expected to require additional insertions involving d zero modes. Note that these extra d zero modes must arise from the regularization of divergences, and cannot be included by simply modifying the gauge-fixing condition as suggested in [7].

This concludes our explanation of the regularization method of [4] and we now turn to the construction of vertex operators in the Siegel gauge.

4 Vertex operators in the Siegel gauge

in the Siegel gauge. In the Siegel gauge, vertices are annihilated by b_0 so the equation of motion implied by $Q = 0$ is simply $L_0 = 0$ (and $\square = 0$ for the massless vertex). Therefore, the gauge is an extension of the Lorentz gauge where the photon wave function satisfies $P^m a_m(x) = 0$.

4.1 Unintegrated vertex operators in the Siegel gauge

In the review of the pure spinor formalism above, we defined the physical vertex operators to be the cohomology elements of the BRST operator

$$Q = Q_0 + Q_1, \quad (4.1)$$

where $Q_0 = \oint \lambda^\alpha d_\alpha$, and $Q_1 = \oint r_\alpha \bar{\omega}^\alpha$. Conventionally, however, the vertex operators have been assumed to be annihilated by Q_0 and Q_1 separately.⁴ This is a particular choice of a gauge for the cohomology representatives of $Q = Q_0 + Q_1$; we shall refer to this gauge as the “minimal gauge”.

In the minimal gauge, vertices are independent of non-minimal variables, $(\bar{\omega}^\alpha, \bar{\lambda}_\alpha, s^\alpha, r_\alpha)$, and have no poles in $\bar{\lambda}\lambda$. For example, the unintegrated vertex operators for the massless and the first massive modes are given by the most general λ -charge 1 operators of this type,

$$V_{\text{massless}}^{\text{min}} = \lambda^\alpha A_\alpha(x, \theta), \quad (4.2)$$

$$V_{\text{1st massive}}^{\text{min}} = \partial \lambda^\alpha B_\alpha(x, \theta) + \lambda^\alpha \partial \theta^\beta B_{\alpha\beta}(x, \theta) + \lambda^\alpha d_\beta B_\alpha{}^\beta(x, \theta) \\ + \lambda^\alpha \Pi^m B_{\alpha m}(x, \theta) + \lambda^\alpha J B_\alpha(x, \theta) + \lambda^\alpha N^{mn} B_{\alpha mn}(x, \theta), \quad (4.3)$$

and it had been explicitly checked that the cohomology condition with respect to $Q_0 = \oint \lambda^\alpha d_\alpha$ yields the correct on-shell constraints and gauge invariance conditions on the superfields $A_\alpha(x, \theta)$, $B_\alpha(x, \theta)$, \dots , $B_{\alpha mn}(x, \theta)$ [1, 29].

Although the minimal gauge is convenient for many calculations, it is more natural and sometimes necessary to allow more general dependencies on the pure spinor sector. A natural class of (massless) vertex operators with ghost number 1 is given by

$$V = V_1 + \dots + V_p \\ = \frac{\bar{\lambda}_{\alpha_0} \lambda^\beta \lambda^\gamma C^{\alpha_0}{}_{\beta\gamma}(x, \theta)}{(\lambda \bar{\lambda})} + \dots + \frac{\bar{\lambda}_{\alpha_0} r_{\alpha_1} \dots r_{\alpha_p} \lambda^\beta \lambda^\gamma C^{\alpha_0 \alpha_1 \dots \alpha_p}{}_{\beta\gamma}(x, \theta)}{(\lambda \bar{\lambda})^{p+1}}, \quad (4.4)$$

where the ghost number is measured by $J_g = \omega \lambda - \bar{\omega} \bar{\lambda}$. Note that as in the b ghost, V has been defined such that $(r_\alpha \frac{\partial}{\partial r_\alpha} + \bar{\lambda}_\alpha \frac{\partial}{\partial \bar{\lambda}_\alpha})V = 0$. Since the composite b -ghost depends non-trivially on $r/(\lambda \bar{\lambda})$, it is clear that the Siegel gauge condition $b_0 = 0$ can be achieved only if one allows the vertex operators to depend on the non-minimal fields as in (4.4).

Note that the choice of (4.4) allows one to choose different gauges for the superfields C ’s on different coordinate patches of the pure spinor space. The simple form of the vertex in (4.2) should then be understood as a special choice of the gauge for C ’s such that the vertex is globally defined on the pure spinor space (or in other words, independent of the non-minimal variables).

We shall now show that the Siegel gauge can be achieved within the general form of the vertices (4.4) by explicitly constructing them. To be concrete, we explain our construction using the massless vertex as an example, but the construction works for massive fields as

⁴ In the Čech type formulation of curved $\beta\gamma$ systems, this corresponds to the operators defined globally on the pure spinor space.

well.⁵ We start from the ghost number 2 cohomology of Q

$$V^* = \lambda^\alpha \lambda^\beta A_{\alpha\beta}(x, \theta), \quad (4.5)$$

with $A_{\alpha\beta}(x, \theta)$ a *bosonic* superfield. Our Siegel gauge vertex operator is then defined as

$$V(z) = b_0 V^*(z) \equiv \oint dy (y - z) b(y) V^*(z) \quad (4.6)$$

which is obviously annihilated by b_0 . More explicitly, V reads

$$V = V_0 + V_1 + V_2 + V_3, \quad (4.7)$$

where

$$V_0 = (b_0)_0 V^* = -\frac{\bar{\lambda}_\alpha \lambda^\beta \lambda^\gamma (\not{p} D)^\alpha A_{\beta\gamma}}{2(\bar{\lambda}\lambda)}, \quad (4.8)$$

$$V_1 = (b_1)_0 V^* = \frac{(\bar{\lambda}\gamma^{mnp}r)[\lambda^\beta \lambda^\gamma (D\gamma_{mnp}D) - 24(\gamma_{mn}\lambda)^\beta \lambda^\gamma P_p]A_{\beta\gamma}}{192(\bar{\lambda}\lambda)^2}, \quad (4.9)$$

$$V_2 = (b_2)_0 V^* = -\frac{\bar{\lambda}_\alpha (r\gamma^{mnp}r)(\gamma_{mn}\lambda)^\beta \lambda^\gamma (\gamma_p D)^\alpha A_{\beta\gamma}}{16(\bar{\lambda}\lambda)^3}, \quad (4.10)$$

$$V_3 = (b_3)_0 V^* = \frac{(\bar{\lambda}\gamma^{mnp}r)(r\gamma_p{}^{qr}r)(\gamma_{mn}\lambda)^\beta (\gamma_{qr}\lambda)^\gamma A_{\beta\gamma}}{256(\bar{\lambda}\lambda)^4}. \quad (4.11)$$

Let us now study the gauge condition implied by the construction above. The cohomology condition on V^* implies that $A_{\alpha\beta}(x, \theta)$ is subject to the constraint and gauge invariance of

$$D_{((\alpha} A_{\beta\gamma))} = 0, \quad \delta A_{\alpha\beta} = D_{((\alpha} \Omega_{\beta))}, \quad (4.12)$$

where $\Omega_\beta(x, \theta)$ is an arbitrary superfield, and the notation $((\alpha_1 \cdots \alpha_n))$ signifies symmetric γ -traceless combination of spinorial indices. From the antifield calculus in ten dimensional super-Maxwell theory, it is well known [2, 30] that $A_{\alpha\beta}(x, \theta)$ contains a vector at θ^4 , whose “equation of motion” is the Lorentz gauge condition

$$P^m a_m(x) = 0. \quad (4.13)$$

So our construction explains that the Siegel gauge $b_0 V = 0$ is indeed an extension of the Lorentz gauge, as is expected.

Several remarks are in order before turning to the construction of integrated vertex operators in the Siegel gauge.

⁵ Antifield vertices for the massive modes have not been computed explicitly in the pure spinor literatures, but strong evidence for the fact that the space of pure spinor vertices enjoys field-antifield symmetry was presented in [23, 24].

First, we note that the construction above in fact parallels that of the bosonic string. In the bosonic string, the notion of the Siegel gauge and that of field ($V = c\psi(x)$) and antifield ($V^* = c\partial c\psi^*(x)$) are manifestly related because there the field-antifield doubling comes from the ghost zero-mode oscillators satisfying $\{b_0, c_0\} = 1$. Also, it is clear that the field (or Siegel gauge) vertex operators can be obtained from the antifield vertex operators by acting with b_0 .

In the pure spinor formalism, however, there is no c ghost so *a priori* the field-antifield doubling and the Siegel gauge choice are unrelated. Moreover, the b ghost is a complicated operator that may have non-trivial cohomologies, so one might even worry that the condition $b_0V = 0$ on the vertex of the form (4.4) does not have a solution. However, our construction explains that the only structure needed for solving $b_0V = 0$ is the field-antifield symmetry of the operator space. The presence of the field-antifield symmetry is non-trivial in the pure spinor formalism, but is strongly supported by the study of the pure spinor partition functions in [23, 24].

Finally, in our construction, we have not specified the gauge for the superfield in V^* . However, it is easy to see that any choice of gauge leads to a vertex in the Siegel gauge. The “pre-gauge transformation” $\delta V^* = Q(\lambda^\alpha \Omega_\alpha)$ simply modifies the Siegel gauge vertex V by

$$\delta V = b_0(Q(\lambda^\alpha \Omega_\alpha)) = L_0(\lambda^\alpha \Omega_\alpha) - Q(b_0(\lambda^\alpha \Omega_\alpha)) \quad (4.14)$$

The first term vanishes if Ω_α has weight 0, and the second term is the remaining gauge transformation in Siegel gauge analogous to the residual gauge transformation of the Maxwell theory in the Lorentz gauge.

4.2 Massless integrated vertex operator in the Siegel gauge

We now turn to the construction of the integrated vertex operators. We exclusively consider massless vertices. Given a Q -closed unintegrated vertex operator V in an arbitrary gauge, the corresponding integrated vertex operator U satisfying

$$QU(w) = \partial V(w) \quad (4.15)$$

can be obtained by defining

$$U(w) = b_{-1}V(w). \quad (4.16)$$

Since $\{Q, b_{-1}\} = L_{-1}$, it is clear that U satisfies (4.15).

For the Siegel gauge vertex operator $V = b_0V^*$ of the previous subsection, U is a conformal primary of weight 1. Indeed, since b and V^* has at most a double pole, one easily finds that for $n > 0$,

$$L_n U = b_{n-1}(b_0V^*) + b_{-1}(b_nV^*) = -b_0(b_{n-1}V^*) + 0 = 0. \quad (4.17)$$

Schematically, the integrated vertex operator $U = b_{-1}V$ is of the form

$$U = (b_{-1})_{-1}V + \partial\theta^\alpha f_\alpha + \Pi^m f_m + d_\alpha f^\alpha + \frac{1}{2}N^{mn}f_{mn}, \quad (4.18)$$

where $(b_{-1})_{-1}V$ denotes the simple pole of $(s^\alpha \partial \bar{\lambda}^\alpha)$ with V , and the f 's are constructed from λ^α , $\bar{\lambda}_\alpha$, r_α , and spacetime derivatives of the superfield $A_{\alpha\beta}$, e.g. $D^n A_{\alpha\beta}(x, \theta)$. Since their r dependence and the order of divergence as $(\bar{\lambda}\lambda) \rightarrow 0$ become important for our application, we record them here:

$$f_\alpha = \frac{D^3 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^0} + \frac{r D^2 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^1} + \cdots + \frac{r^3 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^3}, \quad (4.19)$$

$$f_m = \frac{D^4 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^0} + \frac{r D^3 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^1} + \cdots + \frac{r^4 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^4}, \quad (4.20)$$

$$f^\alpha = \frac{D^5 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^0} + \frac{r D^4 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^1} + \cdots + \frac{r^5 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^6}, \quad (4.21)$$

$$f_{mn} = \frac{D^6 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^0} + \frac{r D^5 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^1} + \cdots + \frac{r^6 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^6}. \quad (4.22)$$

Although the vertex operator U appears complicated, it simplifies considerably after using the gauge invariance $\delta A_{\alpha\beta} = D_{(\alpha} A_{\beta)}$ to gauge-fix

$$(\bar{\lambda}\gamma^m)^\alpha A_{\alpha\beta} = 0. \quad (4.23)$$

To see that this gauge choice is accessible, choose a $U(1) \times SU(5)$ decomposition of $SO(10)$ such that the only non-vanishing component of $\bar{\lambda}_\alpha$ carries $-\frac{5}{2}$ $U(1)$ charge. If A_{ab} (for $a = 1$ to 5) denotes the component of $A_{\alpha\beta}$ with $+3$ $U(1)$ charge, the constraint $\lambda^\alpha \lambda^\beta \lambda^\gamma D_\alpha A_{\beta\gamma} = 0$ implies that $D_{(a} A_{bc)} = 0$ where D_a is the component of D_α with $\frac{3}{2}$ $U(1)$ charge. Since $\{D_a, D_b\} = 0$, $D_{(a} A_{bc)} = 0$ implies that $A_{ab} = D_{(a} \Omega_b)$ for some Ω_b . So Ω_b can be used to gauge $A_{ab} = 0$. In the gauge $A_{ab} = 0$, $\lambda^\alpha \lambda^\beta \lambda^\gamma D_\alpha A_{\beta\gamma} = 0$ implies that $D_{(a} A_b^{[cd]} = 0$ where $A_b^{[cd]}$ denotes the component of $A_{\alpha\beta}$ with $+1$ $U(1)$ charge. So $A_b^{[cd]} = D_b \Omega^{[cd]}$ for some $\Omega^{[cd]}$, which means that $A_b^{[cd]}$ can also be gauged to zero. In the gauge where $A_{ab} = A_b^{[cd]} = 0$, it is easy to verify that $(\bar{\lambda}\gamma^m)^\alpha A_{\alpha\beta} = 0$.

Since $(\bar{\lambda}\gamma^m r) = 0$ implies that the $U(1)$ charge of r_α is either $-\frac{1}{2}$ or $-\frac{5}{2}$, one can use $U(1)$ invariance to verify in this gauge that all terms beyond r^3 in $(f_\alpha, f_m, f^\alpha)$ vanish, and that all terms beyond r^4 in f_{mn} vanish. Note that $\lambda^\alpha \lambda^\beta \lambda^\gamma D_\alpha A_{\beta\gamma} = 0$ implies in this gauge that the $U(1)$ charge of $D_\alpha A_{\beta\gamma}$ is less than or equal to $-\frac{3}{2}$. It will turn out that when we compute 4-point 1-loop amplitude using 4 U 's, the only contribution will come from the r^3 term in $d_\alpha f^\alpha$ and the r^4 term in $\frac{1}{2}N^{mn}f_{mn}$, namely

$$d_\alpha \frac{r^3 D^2 A_{\alpha\beta}(x, \theta)}{(\bar{\lambda}\lambda)^3} + N_{mn} \frac{r^4 D^2 A_{\alpha\beta}(x, \theta)}{(\bar{\lambda}\lambda)^4}. \quad (4.24)$$

This concludes our construction of the massless integrated vertex operator in the Siegel gauge, and we now argue that it can be used to compute n -point 1-loop amplitudes using only integrated vertex operators.

5 New n -point 1-loop amplitude prescription

In this section, it will be shown that n -point 1-loop amplitudes in the pure spinor formalism can be computed using n Siegel gauge integrated vertex operators of the previous section.

5.1 Description of the problem

In bosonic string theory, the canonical prescription for computing n -point 1-loop amplitudes is to use 1 unintegrated vertex operator and $n - 1$ integrated operators, with a single insertion of the b ghost:

$$\mathcal{A}_n = \int d^2\tau \int \left(\prod_{i=2}^n d^2w_i \right) \left| \langle \int d^2z (b \cdot \mu)(z) V(w_1) \prod_{i=2}^n U(w_i) \rangle \right|^2. \quad (5.1)$$

However, it is well known that when the vertex operators are in Siegel gauge, the amplitude can also be computed using only the integrated vertex operators as in

$$\mathcal{A}_n = \int \frac{d^2\tau}{\text{Im } \tau} \int \left(\prod_{i=1}^n d^2w_i \right) \left| \langle J_g(z) \prod_{i=1}^n U(w_i) \rangle \right|^2. \quad (5.2)$$

Here $J_g = -bc$ is the ghost number current (put at an arbitrary point z on the worldsheet). So a natural question is if a similar prescription can also be used in the pure spinor formalism when the vertex operators are in Siegel gauge.

To understand why Siegel gauge is necessary, let us first explain why prescriptions of the type (5.2) with integrated vertex operators in the minimal gauge ($Q_1 = 0$) give zero for the massless 4-point 1-loop amplitude. In the non-minimal formalism, the bosonic prescription (5.2) naively generalizes to

$$\mathcal{A}_n = \int \frac{d^2\tau}{\text{Im } \tau} \int d^2w_1 \cdots d^2w_n \left| \langle \mathcal{N} J_g(z) U(w_1) \cdots U(w_n) \rangle \right|^2, \quad (5.3)$$

where $J_g = \omega\lambda - \bar{\omega}\bar{\lambda}$ is the ghost number current defined so that the BRST charge Q carries charge +1, and

$$\mathcal{N}_0 = \exp[-(\bar{\lambda}\lambda + r\theta) - (\frac{1}{2}\bar{N}^{mn}N_{mn} + \bar{J}_{\bar{\lambda}}J_{\lambda} + \frac{1}{2}S^{mn}(\lambda\gamma_{mn}d) + S(\lambda d)] \quad (5.4)$$

is the zero-mode regularization factor that is needed to define an indefinite factor $(\infty \cdot 0)$ coming from non-compact bosonic integrals and unsaturated fermionic integrals.

Now, in order to have a non-vanishing result, one must saturate the 16 zero-modes of d_α on the torus. However, \mathcal{N}_0 can provide at most 11 d_α zero-modes, and each unintegrated vertex operator can only provide 1 d_α zero-mode. So, for the 4-point amplitude it is impossible to saturate the d_α zero-modes in (5.3) and one gets a vanishing result.

To have a non-vanishing 4-point 1-loop amplitude using 4 integrated vertex operators, an additional d_α zero-mode must be supplied from somewhere. In [4] it was suggested that the extra d_α zero-mode could be provided from the additional regulator $\mathcal{N}'(y)$ of section 3, that is needed when the total $\lambda\bar{\lambda}$ pole in the integrand adds up greater than or equal to 11.

Below, we shall show that in Siegel gauge, the 1-loop prescription of the form (5.1) with a b -ghost insertion can be converted to the prescription of the form (5.2) that uses only the integrated vertex operators. Moreover, we shall show that the additional regulator of [4] does provide the missing d_α zero-mode so that the 4-point 1-loop amplitude with this new prescription is non-vanishing.

5.2 The new 1-loop prescription and its derivation

In this subsection, we shall argue that the n -point 1-loop amplitude can be computed by the prescription of the form

$$\mathcal{A}_n = \int \frac{d^2\tau}{\text{Im } \tau} \left(\prod_{i=1}^n \int d^2w_i \right) \left| \langle \mathcal{N}_0 \mathcal{N}'_0 \oint_A dz J_g(z) \left(\prod_{i=1}^n U'(w_i) \right) \rangle \right|^2, \quad (5.5)$$

where \mathcal{N}_0 and \mathcal{N}'_0 are the zero mode regularization factors reviewed above, $J_g = \omega\lambda - \bar{\omega}\bar{\lambda}$ is the ghost number current, and U' is the “smeared version” of the Siegel gauge integrated vertex operator. The smearing was caused by the non-zero modes in $\mathcal{N}'(y)$. Below, we shall omit the prime (that denotes the smearing) from various operators with its presence understood.

We start from the conventional prescription of the form

$$\mathcal{A}_n = \int d^2\tau \left(\prod_{i=2}^n \int d^2w_i \right) \left| \langle \mathcal{N}_0 \mathcal{N}'_0 \oint_A dz (b \cdot \Delta v)(z) V(w_1) \left(\prod_{i=2}^n U(w_i) \right) \rangle \right|^2. \quad (5.6)$$

Here, V is the Siegel gauge unintegrated vertex $V = b_0 V^*$ and $U = b_{-1} V$; a non-trivial cycle A and the discontinuity Δv^z across A of a quasi-conformal vector field v^z are defined in a pair; we take A as a horizontal cycle of length 1 on the real axis, and

$$v^z = \frac{1}{(2i \text{Im } \tau)} (z - \bar{z}) \quad (5.7)$$

has a unit discontinuity across A . v^z is related to the Beltrami differential as $\mu_z^z = \partial_z v^z$.

Since U has no poles with the b ghost (as is the case in the bosonic string), use of this canonical prescription is natural. Moreover, the prescription (5.6) has the full BRST invariance so, barring the usual concern with the moduli boundary contribution, arbitrary BRST trivial pieces may be added to the vertex operators. In particular, one can go to the minimal gauge ($Q_1 = 0$) and there the prescription is well-tested to give the correct answers.

To convert the unintegrated vertex $V(w_1)$ in (5.6) to an integrated one, we first average over its position w_1 :

$$\mathcal{A}_n = \int \frac{d^2\tau}{\text{Im } \tau} \left(\prod_{i=1}^n \int d^2 w_i \right) \left| \langle \mathcal{N}_0 \mathcal{N}'_0 \oint_A dz b(z) V(w_1) \left(\prod_{i=2}^n U(w_i) \right) \rangle \right|^2. \quad (5.8)$$

Note that $\text{Im } \tau$ is the area of the torus of modulus τ . If we were dealing with the bosonic string, a zero-mode of c ghost can be split off from the unintegrated vertex, $V = cU$, and (5.2) is essentially derived. However, in the pure spinor formalism, there is no c ghost so we wish to use the b ghost present in (5.8) to convert V to $U = b_{-1}V$. Therefore we rewrite

$$\oint_A dz b(z) = - \oint_C dz' \oint_A dz b(z') J_g(z) \quad (5.9)$$

where C is a contour that surrounds z . Then, pulling the contour C off z , we get

$$\mathcal{A} = \int \frac{d^2\tau}{\text{Im } \tau} \int \left(\prod_{i=1}^n d^2 w_i \right) \left| \langle \mathcal{N}_0 \mathcal{N}'_0 \oint_A dz J_g(z) \prod_{i=1}^n U(w_i) \rangle \right|^2 \quad (5.10)$$

where we have used that b has no poles with U . This is our prescription for the n -point 1-loop amplitudes that treats all external vertices equally.

Note that (5.10) is valid only for the vertices in a special class of gauges, because not all BRST trivial operators decouple anymore. This explains why the minimal gauge vertices cannot be used in (5.10). However, (5.10) still has a residual gauge invariance since operators of the form $Q(b_0\Omega)$ decouple.

The derivation here of course applies to the bosonic string as well. There, since b and c in J_g contribute only the zero-modes, the integration of $J_g(z)$ over A can be undone,

$$\oint_A dz J_g(z) = -bc(y), \quad (y: \text{arbitrary point}) \quad (5.11)$$

and hence,

$$\mathcal{A} = \int \frac{d^2\tau}{\text{Im } \tau} \int \left(\prod_{i=1}^n d^2 w_i \right) \left| \langle J_g(y) \prod_{i=1}^n U(w_i) \rangle \right|^2 \quad (5.12)$$

as is well known [31]. However, we note again that integrated vertices in this formula are no longer allowed to be in an arbitrary gauge; they must be related to the conventional representatives (e.g. $U = e^{ik \cdot x}$ for the tachyon) by a gauge transformation of the type $\delta U = Q(b_0 \Omega)$.

5.3 Residual BRST invariance of the new prescription

In the derivation of this prescription, it was important that the integrated vertex U was annihilated by b_{-1} . Therefore, it seems that one is no longer free to choose an arbitrary gauge for vertices by adding BRST trivial pieces. However, it will now be argued that, up to a possible contribution from the boundary of the moduli space, U still has a residual gauge invariance of the form

$$\delta U = Q(b_0 \Omega) \quad (5.13)$$

where Ω is an arbitrary weight 1 primary operator.

To show that operators of the form $Q(b_0 \Omega)$ decouple from the amplitude, consider a variation of an $(n+1)$ -point amplitude:

$$\delta \mathcal{A} = \left\langle \oint_A J_g U_1 \cdots U_n Q(b_0 \Omega) \right\rangle. \quad (5.14)$$

Here $U_i \equiv U_i(w_i)$ and we omitted the integrations over w_i 's. Since the U_i 's are Q -closed under the integration symbol, we treat them as if they are Q -closed.

Now, pulling the contour of Q and b_0 off of Ω and using that $Q(U_i) = b_0(U_i) = 0$, we find

$$\delta \mathcal{A} = - \left\langle \oint_A J_B U_1 \cdots U_n b_0 \Omega \right\rangle \quad (5.15)$$

$$= \left\langle \oint_A b_0(J_B) U_1 \cdots U_n \Omega \right\rangle \quad (5.16)$$

$$= \left\langle \oint_A L_0 U_1 \cdots U_n \Omega \right\rangle \quad (5.17)$$

where J_B denotes the BRST current. Note that on a torus, $b_0(X)$ can be written as $[\oint_A b, X]$ so that $b_0(X_1 X_2) = b_0(X_1) X_2 \pm X_1 b_0(X_2)$. Since all vertex operators are primary fields, insertion of $\oint_A L_0$ generates a total derivative on the moduli space. so $Q(b_0 \Omega)$ indeed decouples from the amplitude.

5.4 4-point 1-loop massless amplitude

It will now be shown that if one uses the integrated vertex operator U in Siegel gauge. the new prescription of section 5.2 gives a non-vanishing result for the 4-point 1-loop

massless amplitude. Below, we only write the chiral half of the closed string and use the terminology appropriate for the open string.

We first show that the only non-vanishing contribution comes from the four product of the $d_\alpha r^3$ and $N_{mn} r^4$ terms

$$d_\alpha \frac{r^3 D^2 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^3} + N_{mn} \frac{r^4 D^2 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^4} \quad (5.18)$$

in U . Note that these get smeared to

$$d_\alpha \frac{(r + \bar{g})^3 D^2 A_{\alpha\beta}(x, \theta + g)}{|\lambda + f|^6} + N_{mn} \frac{(r + \bar{g})^4 D^2 A_{\alpha\beta}(x, \theta + g)}{|\lambda + f|^8} \quad (5.19)$$

in the presence of the extra regulator $\mathcal{N}'(y)$ of section 3. The regularization of $(\bar{\lambda}\lambda)^{-L}$ pole at the same time shifts r^L to $(r + \bar{g})^L$, so combinations with $L > 11$ can give non-zero contribution.

To show that these are the only contributions to the amplitude, first recall that, for each of the products in $(\int U)^4$, only one of the two regularization factors \mathcal{N}_0 and \mathcal{N}'_0 is necessary. The former is needed when the total order of $(\bar{\lambda}\lambda)$ poles (or equivalently, the total r -degree) is below $(\bar{\lambda}\lambda)^{-11}$, and the latter is needed when it exceeds $(\bar{\lambda}\lambda)^{-11}$.⁶

Since the regulator

$$\mathcal{N}_0 = \exp[-(\bar{\lambda}\lambda + r\theta) - (\bar{N}N + (\bar{\lambda}\lambda)sd)] \quad (5.20)$$

can only provide at most 11 d_α zero modes, and 4 U 's can provide at most 4 d_α zero modes, it is clear that combinations of the terms for which the total order of $(\bar{\lambda}\lambda)$ pole L is below 11 cannot contribute. Therefore, we can forget about the terms requiring \mathcal{N}_0 regularization (and \mathcal{N}_0 itself).

For combinations of the terms requiring the \mathcal{N}' regularization, one has to saturate the fermionic zero modes of $(d_\alpha, \theta^\alpha)$, (s_α, r^α) and $(g^\alpha, \bar{g}_\alpha)$ to have a non-vanishing result. Unlike the \mathcal{N} regulator of (5.20), the zero mode remnant of the \mathcal{N}' regulator

$$\mathcal{N}'_0 = \exp[-(\bar{\omega}\omega + sd) + (f\omega + gd) + (\bar{f}\bar{\omega} + \bar{g}s)] \quad (5.21)$$

can provide more than 11 d_α zero modes because the zero modes of d_α appear both in $\exp(-sd)$ and $\exp(gd)$. We will find that \mathcal{N}'_0 can provide L d_α zero modes for the combination of terms that goes as $r^L/(\bar{\lambda}\lambda)^L$ in the absence of \mathcal{N}' regularization.

We first show that the r^{12} term in $(d_\alpha f^\alpha)^4$ can saturate all the zero-modes. In the gauge $(\bar{\lambda}\gamma^m)^\alpha A_{\alpha\beta} = 0$, f^α contains terms up to $r^3/(\bar{\lambda}\lambda)^3$, so $(d_\alpha f^\alpha)^4$ contains terms up to

⁶ For the eleventh pole $r^{11}/(\bar{\lambda}\lambda)^{11}$ see the discussion at the end of section 3.2. However, since this r^{11} term cannot saturate the fermionic zero modes for the 4-point 1-loop amplitudes, one may ignore this subtlety here.

$r^{12}/(\bar{\lambda}\lambda)^{12}$. However, as explained just above, terms with total r -degree below 11 cannot saturate d_α zero modes, so we only need to keep $r^{11}/(\bar{\lambda}\lambda)^{11}$ and $r^{12}/(\bar{\lambda}\lambda)^{12}$.

The coefficients of $(\bar{\lambda}\lambda)^{-L}$ ($L = 11$ or 12) in these combinations are

$$e^{-(\bar{\omega}\omega+sd)}e^{(f\omega+gd)+(\bar{f}\bar{\omega}+\bar{g}s)} \times (d)^4(r+\bar{g})^L D^{20-L} A_{\alpha\beta}^4(x, \theta + g) \quad (5.22)$$

where the exponential factors come from the regulator \mathcal{N}'_0 , and the rest come from the smeared vertices (5.19). Now, to saturate all the zero modes of r_α , s^α and \bar{g}_α , one has to take the following combination in (5.22):

$$\begin{aligned} & e^{-\bar{\omega}\omega}e^{(f\omega+gd)+\bar{f}\bar{\omega}}(sd)^{L-11}(\bar{g}s)^{22-L} \times (d)^4(r^{11}\bar{g}^{L-11})D^{20-L}A_{\alpha\beta}^4(x, \theta + g) \\ & = e^{-\bar{\omega}\omega}e^{(f\omega+gd)+\bar{f}\bar{\omega}}d^{L-7}(sr)^{11}\bar{g}^{11}D^{20-L}A_{\alpha\beta}^4(x, \theta + g). \end{aligned} \quad (5.23)$$

Then, it is clear that L has to be 12 in order to saturate the 16 d_α zero modes and 11 g^α zero modes as in

$$\begin{aligned} & e^{-\bar{\omega}\omega}e^{f\omega+\bar{f}\bar{\omega}}(gd)^{11}d^{L-7}(sr)^{11}\bar{g}^{11}D^{20-L}A_{\alpha\beta}^4(x, \theta + g) \\ & = e^{-\bar{\omega}\omega}e^{f\omega+\bar{f}\bar{\omega}}d^{L+4}(sr)^{11}(g\bar{g})^{11}D^{20-L}A_{\alpha\beta}^4(x, \theta). \end{aligned} \quad (5.24)$$

Thus, in $(d_\alpha f^\alpha)^4$, only

$$\left(d_\alpha \frac{r^3 D^2 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^3}\right)^4 \quad (5.25)$$

contributes to the amplitude.

This counting of the zero modes at the same time explains that the \mathcal{N}'_0 regulator can provide L d_α zero modes for the term that naively goes like $r^L/(\bar{\lambda}\lambda)^L$. Then, it is clear that only

$$\left(d_\alpha \frac{r^3 D^2 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^3} + N_{mn} \frac{r^4 D^2 A_{\alpha\beta}}{(\bar{\lambda}\lambda)^4}\right)^4 \quad (5.26)$$

contributes to the 4-point amplitude in the $(\bar{\lambda}\gamma^m)^\alpha A_{\alpha\beta} = 0$ gauge. (Other combinations cannot saturate the 16 d_α zero modes, because they are of the form $d^k r^{L-k}/(\bar{\lambda}\lambda)^{L-k}$ with $L < 16$ and $k = 1, \dots, 4$.)

Let us make a consistency check for the amplitudes computed using the prescription given here. The 4 point amplitude should have the dimension

$$\mathcal{A} \sim F^4, \quad (5.27)$$

where $F = F_{mn}$ is the photon field strengths. In the superfield $A_{\alpha\beta}(x, \theta)$, the fieldstrength resides at the θ^6 level so using $A_{\alpha\beta}$, the amplitude should be

$$\mathcal{A} \sim D^{24} A_{\alpha\beta}^4 \sim \int d^{16}\theta D^8 A_{\alpha\beta}^4, \quad (5.28)$$

and this is what we get by integrating (5.24) over all zero modes (for $L = 12$).

We now show that the amplitude computed as above is indeed BRST invariant. To show that one is computing a BRST invariant quantity, one has to check that the result is invariant under the BRST variation of the regulator,

$$\begin{aligned} \mathcal{N}'(y) &= \exp[Q(s^\alpha \omega_\alpha + g^\alpha \omega_\alpha + \bar{g}_\alpha \bar{\omega}^\alpha)] \\ \rightarrow \mathcal{N}'_{c,\varepsilon,\bar{\varepsilon}}(y) &= \exp[Q(cs^\alpha \omega_\alpha + \varepsilon g^\alpha \omega_\alpha + \bar{\varepsilon} \bar{g}_\alpha \bar{\omega}^\alpha)] \\ &= \exp[-c(\bar{\omega}_\alpha \omega^\alpha + s^\alpha d_\alpha) + \varepsilon(f^\alpha \omega_\alpha + g^\alpha d_\alpha) + \bar{\varepsilon}(\bar{f}_\alpha \bar{\omega}^\alpha + \bar{g}_\alpha s^\alpha)], \end{aligned} \quad (5.29)$$

for some constants c , ε , and $\bar{\varepsilon}$.

To check the invariance of (5.25) under this variation of the regulator, one first notes that the zero mode products in (5.23) with $L = 12$ scale as

$$\begin{aligned} &e^{-c(\bar{\omega}\omega + sd)} e^{\varepsilon(f\omega + gd) + \bar{\varepsilon}(\bar{f}\bar{\omega} + \bar{g}s)} \times (d)^4 (r + \bar{\varepsilon}\bar{g})^{12} D^{12} A_{\alpha\beta}^4(x, \theta + \varepsilon g) \\ &= e^{-c(\bar{\omega}\omega)} e^{\varepsilon(f\omega) + \bar{\varepsilon}(\bar{f}\bar{\omega})} (csd)^1 (\varepsilon gd)^{11} (\bar{\varepsilon}\bar{g}s)^{10} \times (d)^4 (\bar{\varepsilon}^1 r^{11} \bar{g}^1) D^{12} A_{\alpha\beta}^4(x, \theta) \\ &= c^1 (\varepsilon\bar{\varepsilon})^{11} e^{-c(\bar{\omega}\omega)} e^{\varepsilon(f\omega) + \bar{\varepsilon}(\bar{f}\bar{\omega})} (g\bar{g})^{11} d^{16} (sr)^{11} D^{12} A_{\alpha\beta}^4(x, \theta). \end{aligned} \quad (5.30)$$

So to show the BRST invariance, one needs to check that the bosonic integrations provide $c^{-1}(\varepsilon\bar{\varepsilon})^{-11}$:

$$\int d^{22}\omega d^{22}\lambda d^{22}f e^{-c(\bar{\omega}\omega) + \varepsilon(f\omega) + \bar{\varepsilon}(\bar{f}\bar{\omega})} \prod_{i=1}^4 \frac{1}{|\lambda + \varepsilon f_i|^6} \sim c^{-1}(\varepsilon\bar{\varepsilon})^{-11}. \quad (5.31)$$

This scaling can be easily shown by performing the change of variables

$$(\omega, \lambda, f) \rightarrow (\omega', \lambda', f') = (c^{1/2}\omega, c^{-1/2}\lambda, \varepsilon c^{-1/2}f) \quad (5.32)$$

so that the integral becomes

$$c^{-1}(\varepsilon\bar{\varepsilon})^{-11} \times \int d^{22}\omega' d^{22}\lambda' d^{22}f' e^{-(\bar{\omega}'\omega') + (f'\omega') + (\bar{f}'\bar{\omega})'} \prod_{i=1}^4 \frac{1}{|\lambda' + f'_i|^6}. \quad (5.33)$$

Similarly, all contributions from (5.26) can be checked to be invariant under the BRST variation of the \mathcal{N}' regulator.

To summarize, we have shown that the 4-point 1-loop amplitude can be computed using 4 integrated vertex operators U in the Siegel gauge. To be able to do so, it was important that the U 's are conformal primaries of weight 1 and are annihilated by b_{-1} . This explains why one could not compute the amplitude using 4 integrated vertex operators in the minimal gauge.

Since the order of $(\bar{\lambda}\lambda)$ poles in $(\int U)^4$ exceeds 11, a regularization for $(\bar{\lambda}\lambda) \rightarrow 0$ was necessary. We used a regularization method proposed in [4] (and explained in section 3) to define the indefinite factor of the form

$$\int d^{22}\lambda d^{11}\frac{r^L}{(\bar{\lambda}\lambda)^L}, \quad (L > 11). \quad (5.34)$$

Moreover, since the Siegel gauge integrated vertex operator takes a relatively simple form in $(\bar{\lambda}\gamma^m)^\alpha A_{\alpha\beta} = 0$ gauge, we were able to identify the combinations of the terms that contribute to the amplitude and their invariance under the BRST variation of the regulator \mathcal{N}' . Our conclusion is that the only contribution comes from the terms (5.26) that require \mathcal{N}' regularization for $(\bar{\lambda}\lambda) \rightarrow 0$.

Finally, let us mention that we have demonstrated (ignoring the gauge invariance $\delta\omega_\alpha = (\gamma^m\lambda)_\alpha\Omega_m$) that the non-zero modes of s^α in the regulator \mathcal{N}' does convert “extra” r_α zero modes above 11 to d_α zero modes as was advocated in [4].

6 Summary

In this paper, we showed how to construct vertex operators in the pure spinor formalism in the Siegel gauge. Unintegrated vertices in the Siegel gauge can be constructed as $V_S = b_0 V^*$, where V^* is the ghost number 2 vertex of the corresponding antifield. Integrated vertices can then be constructed as usual by $\int U_S = \int b_{-1} V_S$.

The construction is not obstructed by the complexity of the b -ghost of the formalism and works for vertices of all mass levels, provided that the space of pure spinor vertices has field-antifield doubling. Although this latter fact is non-trivial in the pure spinor formalism, it is strongly supported by the study of the partition function of the pure spinor operator space in [21, 22, 23].

For the massless states, an explicit form of the antifield vertex operator is $V^* = \lambda^\alpha \lambda^\beta A_{\alpha\beta}(x, \theta)$, and the computation of the Siegel gauge vertices (both unintegrated and integrated) is straightforward. Although the form of the integrated vertex operator U_S is fairly complicated, we showed that in the gauge where $A_{\alpha\beta}$ satisfies $(\bar{\lambda}\gamma^m)^\alpha A_{\alpha\beta} = 0$, the form of U_S simplifies considerably and contains terms only up to $r^4/(\bar{\lambda}\lambda)^4$.

When vertices are in the Siegel gauge, it is well-known in bosonic string theory that the n -point 1-loop amplitude can be computed using n integrated vertex operators. We have shown that this Siegel gauge prescription is also valid in the pure spinor formalism by deriving it from the conventional prescription that uses 1 unintegrated and $(n-1)$ integrated vertex operators.

This new 1-loop prescription provides a good testing ground for the regularization prescription of [4] (reviewed in section 3) for the functional integration region $(\bar{\lambda}\lambda) \sim 0$. This

regularization becomes necessary when the factor of $r/(\bar{\lambda}\lambda)$ in the integrand accumulates to $r^{11}/(\bar{\lambda}\lambda)^{11}$ or higher. Since the Siegel gauge vertex operators have poles in $(\bar{\lambda}\lambda)$, the 4-point 1-loop amplitude already requires this regularization of $(\bar{\lambda}\lambda) \sim 0$. Although we have not worked out the explicit index contractions, we identified the combinations of terms in 4 U_S 's (in the $(\bar{\lambda}\gamma^m)^\alpha A_{\alpha\beta} = 0$ gauge) that can contribute to the amplitude, and argued that they give a well defined quantity.

Note that if one blindly applies the new 1-loop prescription to the “minimal gauge” vertices that do not depend on non-minimal variables, one would get a vanishing 4-point amplitude because of an undersaturation of d_α zero modes. For the Siegel gauge vertices, we observed that the regularization of $(\bar{\lambda}\lambda) \sim 0$ converts the extra factors of r_α 's to d_α zero modes, and the correct saturation of all fermionic zero modes is realized.

There are several possible continuations of the present work. Firstly, it should be possible to complete the computation of the 4-point 1-loop amplitude using the new prescription by working out the index contractions explicitly. Although the regularization prescription of [4] becomes more complicated when one makes it consistent with the pure spinor constraint, the number of terms that contributes to the amplitude should not change and is fairly small.

Secondly, since we now have a method to construct Siegel gauge vertex operators systematically, it might be possible to obtain a gauge fixed action of the cubic open superstring field theory proposed in [6].

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